Theory of Noiseless Subsystems

Lorenza Viola
(LANL, lviola@lanl.gov)

Outline

Noiseless Subsystems (NSs)

- ***What they are −***
  Most general route to the emergence of noise−protected degrees of freedom in open quantum systems.

  Knill, Laflamme, Viola, PRL **84**, 2525 (2000).

- ***What they are good for −***
  Most general structure for noise−protected quantum information storage. Strategies to quantum noise control are understandable in terms of identification/generation of appropriate NSs.

- ***Where do we stand in practice −***
  - Quantum optics, 2 qubits: Preservation of a 1D subspace (singlet) against collective noise [Kwiat et al., *Science* **290**, 498 (2000)]
  - Trapped ions, 2 qubits: Preservation of a 2D subspace against collective dephasing [Kielpinski et al., *Science* **291**, 1013 (2001)]
  - NMR, 3 qubits: Preservation of a 2D subsystem against general collective noise [In progress, MIT Spatial NMR Laboratory]

Noiseless sub−spaces vs sub−systems

Interaction between system S and environment E

$$H_{SE}=\sum_{\alpha} E_{\alpha}^{(S)}\otimes B_{\alpha}^{(E)}$$

Goal: Noiseless one−qubit coding

- *Noiseless subspaces –* Look for a degenerate action of the interaction operators on a subspace of states of S:

$$E_{\alpha}^{(S)}|i_L\rangle=c_{\alpha}|i_L\rangle\,,\quad \forall\,\alpha,\ i=0,1$$

States $|i_L\rangle\langle i_L|$ are pure, preserved states of S.

- *Noiseless subsystems –* Look for a degenerate action of the interaction operators on a given quantum number (‘‘factor’’) of a subspace of states of S:

$$E_{\alpha}^{(S)}(|\lambda_L\rangle\otimes|m_Z\rangle)=c_{\alpha}|\lambda_L\rangle\otimes\Omega_{\alpha}^{(S)}|m_Z\rangle\,,\quad \forall\,\alpha,\ \lambda=0,1$$

States $|\lambda_L\rangle\langle\lambda_L|\otimes\rho^{(Z)}$ need not be pure states of S.
Quantum information stored in the L−subsystem is preserved.

From three symmetric spins ...

System S: Three distinguishable spin ½ particles

$$S = span\{ \{|0\rangle_1, |1\rangle_1\} \otimes \{|0\rangle_2, |1\rangle_2\} \otimes \{|0\rangle_3, |1\rangle_3\} \} \simeq \mathbb{C}^8$$

Environment E couples symmetrically to each spin:

$$E_\alpha^{(S)} = \left(\sigma_\alpha^{(1)} + \sigma_\alpha^{(2)} + \sigma_\alpha^{(3)}\right)/2 = J_\alpha \ , \quad \alpha = x,y,z$$

- There are no noiseless subspaces of *S*.
  Can we identify a protected degree of freedom in *S* ?
- Choose on *S* a basis of joint J^2, J_z−eigenstates:

$$S = S_{3/2} \oplus S_{1/2} \ , \quad dim\ S_J = 4$$

$$S_{1/2} = span\{ |\lambda, j_z\rangle_{1/2} \ | \ \lambda = 0,1; \ j_z = \pm 1/2 \}$$

$$1/2_1 \oplus 1/2_2 \ \to \ 0 \oplus 1/2_3 \ \text{—} \ 1/2 \ \text{—} \ \lambda = 0$$

$$1/2_1 \oplus 1/2_2 \ \to \ 1 \oplus 1/2_3 \ \to \ 1/2 \ \text{—} \ \lambda = 1 ; \quad 3/2$$

Keyword: The J_α have an identity action on the coordinate λ.

... to a noiseless qubit

Write $S_{1/2} \simeq S_L \otimes S_Z$ via $|\lambda, j_z\rangle_{1/2} \simeq |\lambda\rangle_L \otimes |j_z\rangle_Z$

- The action of the noise operators on $S_{1/2}$ takes the form

$$J_\alpha \simeq \mathbf{1}^{(L)} \otimes \Omega_\alpha^{(Z)}, \qquad \Omega_\alpha^{(Z)} \in Mat(2\times 2, \mathbb{C})$$

Ω_α depends on the choice of basis states $\{|\lambda\rangle_L\}$ in S_L.

- An explicit realization:

$$|0\rangle_L \otimes |+1/2\rangle_Z = \frac{1}{\sqrt{3}}\left(|001\rangle + \omega|010\rangle + \omega^2|100\rangle\right)$$

$$|1\rangle_L \otimes |+1/2\rangle_Z = \frac{1}{\sqrt{3}}\left(|001\rangle + \omega^2|010\rangle + \omega|100\rangle\right) \qquad \omega = e^{2\pi i/3}$$

$$|0\rangle_L \otimes |-1/2\rangle_Z = \frac{1}{\sqrt{3}}\left(|110\rangle + \omega|101\rangle + \omega^2|011\rangle\right)$$

$$|1\rangle_L \otimes |-1/2\rangle_Z = \frac{1}{\sqrt{3}}\left(|110\rangle + \omega^2|101\rangle + \omega|011\rangle\right)$$

- The factor S_L supports a noiseless subsystem of S.

S_L is the state space of a noiseless qubit.

But what is a qubit?

"Recognizing a qubit can be trickier than one might think"

Di Vincenzo, Fort. Phys. **48**, 771 (2000).

- Qubits are subsystems − Not necessarily the ‘‘natural’’ ones associated with the physical degrees of freedom − specified by the ‘‘right’’ algebra of observables, *e.g.*

$$\sigma_x^{(L)} = \frac{1}{6}\left(2\ \vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)} + \vec{\sigma}^{(2)}\cdot\vec{\sigma}^{(3)} + \vec{\sigma}^{(3)}\cdot\vec{\sigma}^{(1)} \right) = Ex_{1\Leftrightarrow 2}$$

$$\sigma_y^{(L)} = -\frac{\sqrt{3}}{6}\left(\vec{\sigma}^{(2)}\cdot\vec{\sigma}^{(3)} - \vec{\sigma}^{(3)}\cdot\vec{\sigma}^{(1)} \right)$$

with both $[\sigma_\alpha^{(L)}, \sigma_\beta^{(L)}] = 2\, i\, \epsilon^{\alpha\beta\gamma} \sigma_\gamma^{(L)}$, $\sigma_\alpha^{(L)\,2} = \mathbf{1}^{(L)}$

- What else is needed ? − Appropriate control for effecting both unitary and non−unitary quantum operations on subsystems. [*Dynamical control; Initialization; Read−out*].

Viola, Knill, Laflamme, quant-ph/0101090, JPA (in press).

The interaction algebra

Assume S, E initially uncorrelated. Write interaction as

$$H_{SE} = \sum_\alpha E_\alpha^{(S)} \otimes B_\alpha^{(E)}, \qquad B_\alpha^{(E)} \text{ linearly independent}$$

- Error operators are operators that can occur in the quantum operation effected by the environment:

$$\rho \to \sum_a A_a \rho A_a^+, \qquad \sum_a A_a^+ A_a = \mathbf{1}^{(S)}$$

- The possible errors are in the interaction algebra ***A***:

$$\boldsymbol{A}_\mathbf{1} = span\{\mathbf{1}^{(S)}, E_\alpha^{(S)}\}$$

A = linear span of products of operators in $\boldsymbol{A}_1$

- Example − Interaction algebra of collective noise:

$$\boldsymbol{A}_\mathbf{1} = span\{\mathbf{1}, J_x, J_y, J_z\}$$

$\boldsymbol{A}_C =$ algebra of totally symmetric operators on $\mathbb{C}^8$

$$dim\,\boldsymbol{A}_C = dim\left(\boldsymbol{A}_C^{(J=3/2)} \oplus \boldsymbol{A}_C^{(J=1/2)}\right) = 4^2 + 2^2 = 20$$

Noiseless subsystems

$\boldsymbol{A}$ is an operator algebra closed under Hermitian conjugation.

- *Theorem –* The state space S of S can be represented as

$$S \simeq \oplus_J \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J} \qquad \text{such that}$$

$$\boldsymbol{A} \simeq \oplus_J \mathbf{1}_{n_J} \otimes Mat\left(d_J, \mathbb{C}\right)$$

- Each factor $\mathbb{C}^{n_J}$ is the state space of a n_J−dim NS of S.

  A NS is defined by an irrep of the commutant $\boldsymbol{A}'$ of $\boldsymbol{A}$:

$$\boldsymbol{A}' \simeq \oplus_J Mat\left(n_J, \mathbb{C}\right) \otimes \mathbf{1}_{d_J}$$

- Require $n_J \geqslant 2$ for some J for quantum information encoding: Need non−abelian $\boldsymbol{A}'$. ($\boldsymbol{A}'$ contains the NS−observables!)

- Example − Commutant algebra of collective noise:

  $\boldsymbol{A}'_C$ = algebra generated by the scalars $\left\{ \vec{\sigma}^{(i)} \cdot \vec{\sigma}^{(j)} \right\}_{i \neq j}$

  = natural representation of the group algebra $\mathbb{C}\, S_3$

NSs and Error−Avoiding codes

- A non−trivial $\boldsymbol{A'}$ implies the existence of a non−trivial group of symmetries $G \subseteq U(\boldsymbol{A'})$ such that the overall dynamics is invariant under G.

- Example − Collective noise:

  G = natural representation of the permutation group S_3.

- A noiseless subspace (also: Decoherence−Free Subspace, DFS) occurs in the special case where $d_J = 1$ for some J:

$$\mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J} = \mathbb{C}^{n_J} \otimes \mathbb{C} \simeq \mathbb{C}^{n_J}$$

Zanardi & Rasetti, PRL **79**, 3306 (1997).
Lidar, Chuang, Whaley, PRL **81**, 2594 (1998).

- Noiseless subspaces may be more difficult to find. Protecting one qubit into a noiseless subspace against collective noise requires $n = 4$ symmetric spin ½ particles. NSs are the most general quantum error−avoiding codes.

NSs and Error−Suppression schemes

- Symmetries can be enforced via an active control action.

Viola, Knill, Lloyd, PRL **82**, 2417 (1999).

- In the limit of fast control, the overall effective Hamiltonian takes the symmetry of the control operations:

$$\bar{H}_{eff} = \frac{1}{|G|}\sum_{g_l \in G} g_l^{+} H g_l \in \mathbb{C}G' \qquad \text{G = decoupling group}$$

- Example − Permutation−invariant (collective) dynamics can be synthesized by choosing a decoupling group

G = symmetric (permutation) group, S_n

- The method allows to dynamically generate NSs via

$$S \simeq \oplus_J \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}$$

$$\mathbb{C}G \simeq \oplus_J \mathbf{1}_{n_J} \otimes Mat(d_J, \mathbb{C})$$

$$\mathbb{C}G' \simeq \oplus_J Mat(n_J, \mathbb{C}) \otimes \mathbf{1}_{d_J}$$

Viola, Knill, Lloyd, PRL **85**, 3520 (2000).

QEC for general noise

- *Error analysis* – Evaluate error operators on the basis of the order in time with which they can occur:

$$\rho \rightarrow \quad t\left(E_1^{(1)}\rho E_1^{(1)\,+} + E_2^{(1)}\rho E_2^{(1)\,+} + \ldots\right) + \qquad E_k^{(1)} \in \boldsymbol{A}_1$$
$$t^2\left(E_1^{(2)}\rho E_1^{(2)\,+} + E_2^{(2)}\rho E_2^{(2)\,+} + \ldots\right) + \ldots \quad E_k^{(2)} \in \boldsymbol{A}_2$$

- $\boldsymbol{A_e} = \boldsymbol{A_1}^e$ **=** span of products of *e* or less operators in $\boldsymbol{A_1}$

  $\boldsymbol{A_e}$ contains all error operators of weight (at most) *e*

$$\boldsymbol{A_1} \subseteq \quad \boldsymbol{A_2} \subseteq \quad \ldots \quad \subseteq \quad \boldsymbol{A}$$

  For the independent error model, *e* = ‘‘number of errors’’

- An *e*−Quantum Error−Correcting code is a code that can recover from all error operators in $\boldsymbol{A_e}$.

- Information protected in an *e*−QEC code is affected by noise to order $e + 1$ or higher in time.

NSs vs QEC codes

- An *infinite*−Quantum Error−Correcting code is a code that can recover from all error operators in $\boldsymbol{A}$.
- *Claim* – NS corresponds to infinite−QEC codes.
  Let $S \simeq L \otimes Z \oplus D$, and $E_k\left(|\psi\rangle_L \otimes |s_0\rangle_Z\right) = |\psi\rangle_L \otimes |s_k\rangle_Z \quad \forall E_k, |s_0\rangle_Z$
  $C \simeq L \otimes |s_0\rangle_Z$ satisfies the necessary & sufficient conditions for recovery from all errors in $\boldsymbol{A}$.
  Recovery super−operator $\boldsymbol{R}$ = ''reset'' operation $|s_k\rangle_Z \to |s_0\rangle_Z$
  (C, $\boldsymbol{R}$) is an infinite−QECC.
- The state $|s_0\rangle_Z$ in the ''syndrome subsystem'' Z is arbitrary:
  No recovery needed for quantum information maintenance.
  (L, $\boldsymbol{R} = \boldsymbol{1}$) is an infinite−QECC.
- The three−spin NS is the smallest one−bit infinite−QECC for collective errors.

QEC codes as subsystems

- *Claim* – QEC codes can be represented as subsystems.
  Use ''syndrome characterization'' $S \simeq C \otimes E \oplus D$.

  Knill, Laflamme, PRA **55**, 900 (1997).

- Example – The three−bit repetition code for bit−flip errors:

$$\left(c_0|0\rangle + c_1|1\rangle\right) \otimes |00\rangle \to c_0|0_L\rangle + c_1|1_L\rangle = c_0|000\rangle + c_1|111\rangle$$

$$\{ E_k \} = \{ E_0 = \mathbf{1}, E_1 = X_{1,} E_2 = X_{2,} E_3 = X_3 \}$$

$$V^0 = span\{ |000\rangle, |100\rangle, |010\rangle, |001\rangle \} = span\{ |v_k^0\rangle, k = 0,\ldots,3 \}$$

$$V^1 = span\{ |111\rangle, |011\rangle, |101\rangle, |110\rangle \} = span\{ |v_k^1\rangle, k = 0,\ldots,3 \}$$

Then $S \simeq \mathbb{C}^8 \simeq C \otimes E$ via $|v_k^i\rangle \simeq |i\rangle_C \otimes |s_k\rangle_E$, $|s_k\rangle_E = |k_1 k_2\rangle_E$
with the identification $C \simeq C \otimes |s_0\rangle_E$ ($|s_0\rangle_E = |00\rangle_E$, no error)

Error action: $E_k \simeq \mathbf{1}^{(C)} \otimes |s_k\rangle\langle s_0|^{(E)}$

Required recovery: $R_k \simeq \mathbf{1}^{(C)} \otimes |s_0\rangle\langle s_k|^{(E)}$

QEC codes vs NSs

- C is not a NS of the full interaction algebra ***A***: Repeated errors do affect the information in C, *e.g.* $X_1 X_2 \neq \mathbf{1}^{(C)} \otimes O(1,2)^{(E)}$.
- C is a NS of the algebra generated by $\boldsymbol{A_1 R}$, where $\boldsymbol{R} = \{ R_a \}$ is the reset quantum operation on the ancillae:

$$\ldots E_{b'} R_{a'} E_b R_a \simeq \mathbf{1}^{(C)} \otimes O(a,b,\ldots)^{(E)}$$

- *Theorem –* Every e−error−correcting code arises as a NS of the algebra generated by $\boldsymbol{A_e R}$ for some quantum operation ***R***. Conversely, every NS of such an algebra $\boldsymbol{A_e R}$ corresponds to an e−error−correcting code.

Knill, Laflamme, Viola, PRL **84**, 2525 (2000).

- Special case $e = \infty$: Algebra generated by $\boldsymbol{A_e R} = \boldsymbol{A}$.
  NSs of the full interaction algebra are infinite−QEC codes, and viceversa.

Conclusions

- The notion of a noiseless subsystem provides a unifying conceptual framework for realizing noise control in open quantum systems and quantum information processing.
- Experimental implementations of noiseless subsystems are within the reach of current quantum information technologies.
- Several challenges remain in theory and in practice, *e.g.*:
  - Implications of the NS notion for quantum information theory (Operator rather than state−vector approach?)
  - Implications of NS−encoded qubits for universal quantum computing architectures (Encoded universality)
  - Alternative strategy for characterizing/identifying NSs (Generalized predictability sieve?)
  - Improvement of attainable quantum control and fidelities
  - Implementation of quantum control & quantum logic over one or more NSs

.... *Work in progress* ....

References

Noiseless/Decoherence−Free Subspaces

− P. Zanardi & M. Rasetti, *Phys. Rev. Lett.* **79**, 3306 (1997)
− L.M. Duan & G.C. Guo, *Phys. Rev. Lett.* **79**, 1953 (1997)
− D.A. Lidar, I.L. Chuang, K.B. Whaley, *Phys. Rev. Lett.* **81**, 2594 (1998)
− D.A. Lidar, D. Bacon, K.B. Whaley, *Phys. Rev. Lett.* **82**, 4556 (1999)

Noiseless Subsystems

− E. Knill, R. Laflamme, L. Viola, *Phys. Rev. Lett.* **84**, 2525 (2000)
− P. Zanardi, *Phys. Rev. A* **63**, 012301 (2001)
− J. Kempe *et al., Phys. Rev. A* **63**, 042307 (2001)
− L. Viola, E. Knill, R. Laflamme, quant-ph/010109, *J. Phys. A* (Aug 2001)
− L. Viola, E. Fortunato *et al.,*"E*xperimental realization of noiseless subsystems for quantum information processing*", submitted (2001)
− See also lecture notes at http://www.c3.lanl.gov/~knill/trnsps.html

Quantum Error Suppression and Subsystems

− L. Viola & Lloyd, *Phys. Rev. A* **58**, 2733 (1998)
− L. Viola, E. Knill, S. Lloyd, *Phys. Rev. Lett.* **82**, 2417 (1999)
− L. Viola, S. Lloyd, E. Knill, *Phys. Rev. Lett.* **83**, 4888 (1999)
− L. Viola, E. Knill, S. Lloyd, *Phys. Rev. Lett.* **85**, 3520 (2000)